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Abstract

A survey is made of various singular limits of solutions of the Boltzmann equation which, taken together, describe most of the qualitative and quantitative properties of a rarefied gas.* Included are large and small mean-free-path limits, boundary and initial layers, and the influence of instabilities on the propagation of correlations by non-chaotic Boltzmann equations.

Introduction

The study of singular solutions is a singular preoccupation of applied mathematicians. This is not entirely due to an aberrant interest in mathematical pathology but has a rational foundation. A physical problem will usually contain a number of parameters. One can normally expect (and as part of a strong existence theorem, prove) continuous dependence of the solution on parameters. If a parameter has a natural range (eg. $0 < \lambda < \infty$), then one will almost automatically find singular behavior at the endpoints. A detailed, quantitative description of the singular endpoints together with continuous dependence in between will

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* A more detailed survey along similar lines was presented in a Symposium in New York City in April, 1967 and published in Ref. [1].

usually give a very good qualitative picture of the entire problem. One can even say without exaggeration that in a large part of theoretical physics it is a singular approximation which is the physical model, not the underlying general formulation (cf. the "WKB" expansion of a solution of the Schrödinger equation).

It is interesting to compare the analytical attraction to singular limits with the corresponding numerical phobia. Numerical calculations are normally feasible only for intermediate parameter values which are not too close to the singular endpoints. There is therefore a natural symbiotic relation between analytical and numerical analysis.

Singular Limits

The first question to ask is, where are the singular limits in the theory of the Boltzmann equation? The answer is, everywhere! The argument of the equation, in a simple gas, is a molecular density distribution, $f(v,x,t)$. Introducing non-dimensional variables in the equation yields a dimensionless factor K, the Knudsen number; if there is an external velocity for comparison, there is a second factor, the Mach number, M. There is singular behavior for small and large extremes of each of the quantities (v,x,t,K,M). Some are obviously singular such as small x (boundary layer near a wall) or small K (transition from kinetic theory to fluid dynamics at small mean free path); others are not so evidently singular such as large t (decay to equilibrium) or large K (near free flow) or small or large v - but all are in fact singular.

Combinations of singular limits, two or three at a time, are almost virgin territory but are a necessary prerequisite to proper qualitative understanding of the theory of the Boltzmann equation. As one example, a coordinated limit in which $M \to \infty$, $K \to \infty$, and cone angle $\to 0$ makes the flow past a cone solvable for the nonlinear, hard sphere Boltzmann equation [2].

In a gas mixture, the concentration and molecular mass ratio are additional parameters with singular endpoints; and in a plasma the number of parameters and singular limits is almost beyond belief [3].

A basic and mysterious formula of kinetic theory is

$$M = KR \tag{1}$$

where R is the Reynold's number. The intuitive independence of viscous effects measured by R, compressibility, measured by M, and rarefaction, measured by K is apparently contradicted by this formula. The simplest reconciliation is shown in Fig. 1, where it is seen that fluid dynamics covers a one-dimensional (or $1 + \varepsilon$ dimensional) range in which there is either compressibility or viscosity, but not both together to any appreciable extent. It will come as no surprise that the macroscopic limit, $K \to 0$, will take two entirely different forms depending on whether the limit is approximately incompressible or approximately inviscid (arrows in Fig. 1).

Sources of Singular Behavior

An important qualitative contribution to understanding many of the singular limits is afforded by the spectrum of the linearized Boltzmann operator. There is a basic conflict between

the symmetric collision operator and the skew-symmetric streaming operator. The former, taken alone has a real spectrum and the latter pure imaginary; together the spectrum is (very!) complex. Moreover, both the collision and streaming operators contribute continua.

To be more concrete, one frequently encounters a collisional decay factor $\exp(-\nu x/v)$ (either in an explicit formula or in a reduction of the Boltzmann equation to an integral equation). Fast molecules may be relatively undamped even for large values of x. This leads, for example, to a long tail upstream of a shock wave from the fast particles originating downstream. In a more extreme case, it leads to runaway electrons.

The collision kernel is another source of nonuniformities. The collision operator is always singular, usually unbounded, and for real potentials (with infinite total cross-section), the collision operator is not yet mathematically identified, even if the intermolecular potential decays very rapidly [4]. Except for the theory of transport coefficients (which is relatively complete), there is only a scattering of results for real (infinite range) potentials [5,6]. But there is good reason to believe that the problem should yield results in at least a few carefully chosen special cases.

Remarks on Asymptotic Expansions

Almost all singular expansions are asymptotic rather than convergent. In addition to the technical mathematical difference, there is an important psychological difference between asymptotic

and convergent expansions which leads to a conflict between our desires and mathematical reality. If the first term in an expansion is accurate, one is ready to stop there; if it is poor, there is a strong temptation to continue. But in an asymptotic expansion the optimum number of terms varies inversely with small expansion parameter. In other words, if the first term is good, more terms will yield a superb result; if the first term is poor, additional terms will probably make it worse. For example, Burnett and higher order approximations in the Chapman-Enskog hierarchy can be expected to improve a Navier-Stokes result where it is already quite accurate; but they should not be expected to penetrate further into the finite mean-free-path regime. Similarly, dense gas kinetic equations can be expected to improve on the Boltzmann equation only where the latter is already quite good. The analogy with the virial expansion in equilibrium is misleading since the latter expansion is convergent.

Another elementary property of asymptotic approximations is that, despite a theorem to the contrary, they are frequently not unique in practice. The reason is that it may be undesirable to write a sequence of asymptotic approximations (eg. iterations) as a canonical power series. For example, a common method of proving that a series is asymptotic to a solution of a differential equation is to construct (by iteration) a convergent sequence which at the n'th iteration agrees (if expanded) with n terms of the expansion; both the series and the iterative sequence are equally asymptotic. This is the key to the distinction between the Hilbert and Chapman-Enskog expansions, each of which is

asymptotic to a true solution of the Boltzmann equation. Hilbert expands the solution as a power series in a parameter, ε. Chapman-Enskog expand the equations for the fluid state. More precisely, the Hilbert solution for the evolution of the fluid state is obtained by solving the inviscid Euler equations with successively higher order inhomogeneous terms, whereas the Chapman-Enskog equations are successively higher order in derivatives of the dependent variables (as well as in ε).

Recall, in Fig. 1, the two approaches to fluid dynamics. At fixed M and $1/R \to 0$, there is a fluid boundary layer of thickness $\varepsilon^{1/2}$ and a kinetic boundary layer of thickness ε. With a coordinated boundary layer and kinetic expansion, Trilling obtained an expansion differing from both Hilbert and Chapman-Enskog in that the lowest order is Navier-Stokes (not Euler), followed by successive inhomogeneous Navier-Stokes systems [7]. The other approach, at fixed R and $M \to 0$, has no fluid boundary layer; viscous effects are global. But again, an appropriately coordinated fluid (Oseen or Stokes) and kinetic scaling gives Navier-Stokes followed by inhomogeneous Navier-Stokes systems [1].

The inhomogeneous Navier-Stokes system is much more practical than either the Hilbert or Chapman-Enskog. Although each expansion is asymptotic to the true solution, the Navier-Stokes has a much better error estimate than the Hilbert, and it has the advantage of being a familiar equation with a known theory as compared to the higher order Chapman-Enskog systems. But a trivial remark shows that one can have his cake and also eat it. No expansion

(Hilbert or Chapman-Enskog or the two of Fig. 1) need be followed slavishly. Any one of these expansions uniquely determines the coefficients in any other. One can simply make the ad hoc decision to keep up to second derivatives of the dependent variables (Navier-Stokes) and iterate on all higher order terms, making them inhomogeneous. This procedure accomplishes three things. It is asymptotic over a larger range than a Hilbert approximation; it has a readily available theory, as compared to higher order Chapman-Enskog; and it resolves the old problem of where to find additional boundary conditions for the higher order Chapman-Enskog equations - they are simply not needed! The boundary conditions will be, in number and quality, those appropriate to the Navier-Stokes system, except that they will be formal power series in ε. A complete formal solution of the linear, steady boundary value problem, following this prescription,has been given [1].

The Ubiquitous Logarithm

Another very elementary mathematical point with profound physical consequences is the "ubiquitous logarithm" [1]. The kinetic theory application is frequently but not exclusively to large K. One often encounters divergent integrals like $\int_1^\infty dr/r$ which become finite on insertion of a convergence factor, say $\phi(\varepsilon r)$, where $\phi(0) = 1$ and ϕ decays rapidly at infinity. An elementary calculation shows that

$$\int_1^\infty \phi(\varepsilon r)dr/r = \log(1/\varepsilon) + 0(1) \qquad (2)$$

The leading term is universal and independent of ϕ; the error term depends on the entire function ϕ. More generally

$$\int_1^\infty \phi(\varepsilon r)\frac{dr}{r^n} = \frac{\phi(0)}{n-1}+\cdots+\frac{\phi^{(n-1)}(0)}{(n-1)!}\varepsilon^{n-1}[\log(1/\varepsilon)+0(1)] \qquad (3)$$

Such an integral arises, typically, when integrating through a boundary layer on two length scales, r and εr. The expansion is local (within the boundary layer) up to and including the coefficient of the logarithm; the error term $\varepsilon^{n-1}0(1)$ is global and will require matching of "inner" and "outer" expansions.

An example of the application of Eq. (2) is the Coulomb potential for which the standard formulas for transport coefficients diverge. The correct leading log term depends on the existence of the phenomenon of Debye shielding, but not on its details. We know before calculation that the Boltzmann equation or the Fokker-Planck equation,either one used with a Debye cutoff, a spherical shielding potential, or a more sophisticated distorted shielding potential must give the same dominant term - only the shielding length (viz. Debye) must be known, and this only within a scale factor.

In ordinary kinetic theory one frequently encounters an expression

$$\int \exp(-\nu x/v)\,\phi\,dx \qquad (4)$$

where ϕ is a free flow expression, known near a body, and the exponential modification enters only at some distance. For example, in heat flow at large K between parallel plates the log enters at the leading term in an expansion in 1/K, between two cylinders at the next term, and between two spheres at the third term. All coefficients up to the log come from explicit local formulas which do not require global solution of the

boundary value problem or matching of boundary layers.

The point to the "ubiquitous logarithm" is that one can predict many qualitative properties with minimal calculation. For example, the fact that there is no virial expansion directly generalizing the Boltzmann equation was predicted on such grounds [8], and was later verified by more detailed calculation [9].

The Approach to Fluid Dynamics

Next we turn to the small mean-free-path limit which has probably been studied more intensively than any other problem in kinetic theory. Following Hilbert, we place a large parameter, $1/\varepsilon = 1/K$, before the collision term and expand the solution in ε. This expansion is clearly singular since ε multiplies all derivatives. The results confirm the singularity; making no assumption other than that f is expandable in ε leads to a solution which is fully determined by the macroscopic fluid state. This fluid determinism is a virtue if one is interested only in fluid dynamics; it is a vice if one wishes to look more deeply. Only the virtue of the expansion was recognized for a long historical period.

For simplicity consider the Boltzmann equation linearized about an absolute Maxwellian. The perturbed distribution can be projected into orthogonal subspaces,

$$f = \hat{f} + \bar{f} \tag{5}$$

where $\hat{f}$ is the projection into the finite dimensional fluid state, and $\bar{f}$ is the non-fluid complement. The Hilbert and Chapman-Enskog formal expansions can be summarized as yielding

an algorithm in which the fluid component, $\hat{f}$, determines a unique total state f_H as a formal power series in ε,

$$\hat{f} \to f_H(\varepsilon). \tag{6}$$

A more refined analysis of the low ε limit yields a complementary algorithm in which $\overline{f}$ determines a unique total state f_G which decays to zero in a time $t = 0(\varepsilon)$ [10],

$$\overline{f} \to f_G(\varepsilon). \tag{7}$$

Each of the formal expansions, f_H and f_G, individually satisfies the Boltzmann equation term by term. Together they allow the resolution of an arbitrary (smooth) initial f which does not contain ε,*

$$f_0 = f_H(\varepsilon) + f_G(\varepsilon). \tag{8}$$

The prescription $f_0 \to f_H(\varepsilon)$ (which requires calculation of both f_G and f_H) gives the <u>asymptotic</u> fluid initial state f_H which must be applied to the hierarchy of fluid equations to make $f_H(t)$ approximate the true solution of the Boltzmann equation with initial value f_0. For example, one can prove

$$\lim \frac{1}{\varepsilon}|f_{NS}-f| = 0, \quad 0<t_0<t<\infty \tag{9}$$

where f is a solution of the Boltzmann equation, and f_{NS} is a Navier-Stokes solution with suitably modified initial values (<u>not</u> the actual initial fluid state). By adding an appropriate f_G to f_{NS}, the result will be asymptotic uniformly in time, $0 \leq t < \infty$.

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* This resolution, proved rigorously in Refs. [10] and [11], is equivalent to the abstract projection described by W. C. Schieve in these proceedings.

The small mean-free-path limit in a boundary value problem is much more difficult, primarily because the boundary layer is more singular than the initial layer as a consequence of the continuous spectrum. Very sophisticated general existence theorems have been developed by Pao [12], Cercignani [13], and most extensively by Guiraud [14]. But to be applicable to the continuum limit, the estimates must be uniform in the singular limit $\varepsilon \to 0$, and strong estimates are also required for the derivatives of successively higher order that appear in the Chapman-Enskog theory. Such strong estimates have not yet been made in the boundary value problem. Nevertheless, a complete formal theory with algorithms for the asymptotic boundary conditions for Navier-Stokes and higher order approximations, has been given [1], as well as explicit evaluations of many of the higher order slip coefficients, mostly for relaxation models of the Boltzmann equation.* Despite the greater difficulty, one can expect the precision of the theory of the boundary value problem to soon approach that of the initial value problem.

For the nonlinear Boltzmann equation, where the theory is naturally more difficult, there are strong existence theorems only for the physically trivial space-independent initial value problem [15,16]. With a boundary, the nonlinear problem is somewhat amenable to analysis in one space dimension [12]. The three-dimensional initial value problem allows rigorous treatment of the small mean free path limit for weak nonlinearity [16]. Further examination shows, moreover, that the restriction to

*The results, many due to Y. Sone, are scattered through the literature; a list is given in Ref. [1].

weak nonlinearity is to some extent an essential one and not simply a weakness in mathematical technique. We can interpret the proof of convergence to fluid dynamics in the limit of small mean free-path as a very strong stability theorem. A large change in the nonfluid part, $\overline{f}$, of the initial distribution will produce a change of order ε in the total solution, f, at a later time. However, if the fluid dynamics problem is itself unstable (eg. turbulent), then a change in the initial $\overline{f}$ can be reflected in a large change in the subsequent fluid state. In other words,the evolution of the fluid state does not become almost independent of the non-fluid component in the limit of small mean-free-path. Even a very mild instability, such as is exhibited by a non-linear periodic wave whose period depends on its amplitude, will destroy any simple convergence in the continuum limit. For example, to predict the fluid state at time $t = 0(1/\varepsilon)$ (including the phase of the wave) requires knowledge of the entire initial distribution function. The decoupling of the non-fluid component $\overline{f}$ at small mean-free-path is overturned by the enhanced sensitivity induced by macroscopic instability. Whether there are similar divergences more serious than a phase shift is an open question; this may not happen since the stabilizing effect (convergence to fluid dynamics) occurs on a faster time scale than the instability. But if such an effect were to exist, it would invalidate the use of the Navier-Stokes equations even at arbitrarily small mean free-path. Such a question has been brought up by Tsugé in a similar, but different

context, viz. whether the Boltzmann equation itself, considered as an approximation to Liouville's equation, is invalid in an unstable situation.*

Chaos, Correlations, and Turbulence

The transition from the n-particle Liouville description to the 1-particle Boltzmann description is probably the most singular of all limits in kinetic theory [8]. The continuum limit in non-equilibrium statistical mechanics is most easily visualized by letting the molecular diameter (i.e. range of intermolecular potential), σ, approach zero as the number density, n, increases. The thermodynamic limit, $n\sigma^3 \sim 1$, keeps the potential energy comparable to kinetic energy as the mean-free-path ($1/n\sigma^2$) approaches zero. The Boltzmann limit, $n\sigma^2 \sim 1$, gives a limiting continuum which is a perfect gas ($n\sigma^3 \to 0$) but in which collisions exert a finite influence.

To show that the Boltzmann equation governs this limiting continuum requires verification of essentially two conditions, binary collisions and molecular chaos. The former, while mathematically far beyond present techniques of proof [8], is an immediate formal consequence of the limit, $n\sigma^3 \to 0$. The chaos condition, on the other hand, is entirely independent of the rarefaction of the gas, although the two are confused throughout the literature, either explicitly or by the tacit assumption that correlations can only enter with dense gas corrections. Independent of any dynamics, it is necessary to verify initial chaos [8]. One cannot depend on collisions to remove correlations, because the

*Ref. [17], also S. Tsugé in these proceedings.

essential purpose of a Boltzmann equation (as distinguished from macroscopic fluid equations) is to describe a gas on a time scale smaller or comparable to a collision time. Further, it is not even true (though frequently stated) that collisions will destroy correlations to create a chaotic situation, the reason being that long range (macroscopic) correlations will persist for a macroscopically long time [18]. The most direct demonstration of this fact is through the existence of a set of macroscopic conservation equations for two-point correlations just as for one-point quantities. The theorem is that, corresponding to any pair of conserved quantities (eg. mass and energy, angular momentum and linear momentum, etc.) there is, in complete generality (for a liquid as well as a gas), a two-point conservation equation.* Long range correlations will not decay, just as a long wavelength sound wave does not decay (except very slowly, on a long viscous time scale). Conservation of long range correlations shows that the Boltzmann equation does not always govern a rarefied gas.

The initial chaos problem has been solved [19] in the sense that, for a very general class of distributions $f^{(n)}(x_1 \ldots x_n)$, the chaos condition, $f^{(2)} \sim f^{(1)}(x_1) f^{(1)}(x_2)$ has been shown to be much wider than independence $f^{(n)} \sim f^{(1)}(x_1) \ldots f^{(1)}(x_n)$, chaos holding with probability one, in a certain sense, as

*C. Sastri and H. Grad, to appear. The two-point density-density equation has appeared in several places, and the momentum-momentum correlation equation for a perfect gas was given by Tsugé, Ref. [17].

$n \to \infty$. In a subsequent analysis,* the (maximum liklihood) condition from which $f^{(2)} \sim f^{(1)}f^{(1)}$ follows was found to be equivalent to the statement that the given (initial)state $f^{(1)}$ corresponds to a "good" experiment. By "good" we mean that the repeated experiments which are implicitly represented by $f^{(n)}$ give the physical observable, $f^{(1)}$, not only as the expected one-particle distribution, but give it with small dispersion. In other words, an experiment which is accurately reproducible for $f^{(1)}$ has an $f^{(2)}$ which is chaotic. As an example of a bad experiment consider two good experiments, each with reproducible initial values, therefore chaotic, therefore governed by the Boltzmann equation. We toss a coin to decide which experiment to perform. The composite a priori probability is non-chaotic, and the resulting solution of Liouville's equation is not governed by the Boltzmann equation. For example, the two experiments can be operating a shock tube with high pressure on one or the other side of the diaphragm; the Liouville solution is a simple average, whereas the two shock flows are nonlinear and their average cannot satisfy the Boltzmann equation.

Granted initial chaos (a "good" experiment), there are physically realistic situations in a rarefied gas in which the Boltzmann equation is obviously not valid. The question is that of the perpetuation of chaos, granted that it is satisfied initially to some approximation by the choice of a reproducible

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*H. Grad, "Long Range Correlations in a Gas," presented at Yeshiva University, Dec. 1, 1966. See Appendix.

initial state. Tsugé has shown [17], by expanding the two particle distribution $f^{(2)}$ in a generalized 13-moment approximation for a classical boundary layer flow, that the two point correlations become unstable and grow at exactly the same transition point as the main flow. In other words, after the instability develops, chaos is lost and the Boltzmann equation becomes invalid (at least in any naive interpretation).

In ordinary turbulence theory, two point correlations are introduced by superposing an a priori probability (i.e. a sample space of repeated experiments) on the Navier-Stokes equations, taken to be exact. But, if one views the Navier-Stokes equations as arising from kinetic theory, this imposition of a second probability structure is, in the first place redundant, and it also introduces questions of logical compatibility, especially compatibility of ad hoc closure assumptions with the underlying mathematical structure.

To be more precise, compare kinetic theory, in which an n-particle description is taken to be exact, and classical fluid dynamics in which the Navier-Stokes equation is the deterministic starting point. Introducing probability into the first problem gives Liouville's equation, deterministic but in a high dimensional space. Imposing a probability onto the fluid state similarly gives a "Super Liouville" or Hopf equation, this time in a function space in which a point is $\rho(x)$, $u(x)$, $p(x)$ [20]. Lower order distributions such as $f^{(1)}$, $f^{(2)}$, etc. require some closure hypotheses, starting from Liouville. Similarly, in fluid dynamics lower order correlations such as $\langle u(x_1)u(x_2)\rangle$ require closure.

But a closure approximation which is appropriate to the Hopf equation may not be compatible with one obtained from the Navier-Stokes equation considered to be descended via kinetic theory from Liouville's equation; e.g. the symmetry of three-particle density correlations, $\langle n_1 n_{23}\rangle = \langle n_{12} n_3\rangle = \langle n_{123}\rangle$, follows from Liouville but not from Hopf. In other words, the need for a kinetic approach in macroscopic turbulence does not stem from the finite mean-free-path or molecular graininess, but from the fact that the basic symmetries are restricted by the descent from a world of particles.

One can generalize the Boltzmann equation to obtain a sequence of two-point, three-point, etc. kinetic equations for a rarefied gas without invoking molecular chaos. (eg. [17,21]). The closure procedure is not trivial since generalizations of Boltzmann's chaos rule are not unique. For example, one can introduce a closure in an equation governing $\partial f^{(2)}/\partial t$ such that the ordinary Boltzmann equation is exactly satisfied by $f^{(1)} = \int f^{(2)}$ even when $f^{(2)}$ is not chaotic; (this unduly restrictive property is enjoyed by many higher order kinetic equations, eg. see [21,22,23,24]). It is possible however, to formulate a set of desireable criteria which make the closure essentially unique.*

It is useful to make a rough distinction between correlations (referring to two or more positions and one time) and fluctuations (two or more times and any number of positions). To extend the Boltzmann equation to include long range non-chaos, or instability, or turbulence requires correlations. To calculate the scattering

*C. Sastri and H. Grad, to appear.

of light requires fluctuations in time as well as space. In practice, the latter does not necessarily include the former. For example, the density autocorrelation, $\langle n(x,t)n(x',t')\rangle$, can be calculated from the fundamental solution of the ordinary (linearized) Boltzmann equation, viz. the distribution at (x,t) which arises from a δ-function at (x_o,t_o). Correlation information, such as that arising from the two-point conservation equations, is, in principle, not obtainable from the simple Boltzmann equation. There seems to be a blind spot in the literature so that even very general formulations (several positions and several times, eg. [21]) are almost always reduced and approximated in order to be able to confirm the time-hallowed Boltzmann equation and Landau-Lifshitz fluid fluctuation formulas [25].

It is worth remarking that it takes at least a four-point, two-time formulation (two points at each time) to recover, as a special case, the simple two-point, one-time correlation theory. Also, information can be lost not only by taking too primitive a description, but also by too simple a closure rule in the collision term.

To summarize the situation, any instability which allows macroscopic "errors" to grow, may allow the deviation between a Navier-Stokes and Boltzmann solution to grow, and may allow the difference between a chaotic Boltzmann and more basic correlated kinetic solution to grow. Two major and generally

unrecognized points are the fact that long range correlations which nullify the ordinary Boltzmann equation are preserved in a rarefied gas, and they can appear gratuitously by growing out of noise, thereby offering new approaches to fully macroscopic turbulence theory; (this effect is distinct from the interference arising in plasmas from short range micro-instabilities). At stake is the development of a new macroscopic correlation theory based on general conservation equations; a new kinetic theory of fully rarefied gases with correlations; new transitional theories connecting the two as in the Chapman-Enskog, or moment, or more rigorous asymptotic procedures; and a new opportunity to replace the ad hoc closure hypotheses of turbulence theory with more securely founded approximations of kinetic distribution functions.

Conclusion

The theory of the simple Boltzmann equation has developed through three stages. First was the perfection of the calculation of transport coefficients and higher order fluid equations via the Hilbert and Chapman-Enskog theories. These are completely formal analyses, replete with mysteries, but eminently practical for the calculation of transport coefficients to be used in fluid equations. The preoccupation with transport coefficients was broken with the recognition that the primary use of the Boltzmann equation is to elucidate finite mean-free-path phenomena, originally by the introduction of ad hoc moment and

polynomial approximations, later with bimodal, half-range, and other specialized approximations to treat initial and boundary value problems. More recently, precision has been developing in two directions: in the precise mathematical solution of special problems (usually singular), and in general, qualitative theory through existence theorems. The ultimate illumination of the nature of the Chapman-Enskog expansion resulted from a combination of these two approaches.

For the simple Boltzmann equation, the future should involve exploitation of the many known techniques to complete the analysis of the host of remaining singularities; development of a general theory for boundary value problems to match that of the initial value problem; and, hopefully, progress into the world of real potentials with infinite total cross-section. The extension of the Boltzmann equation to complex molecules, which has concentrated largely on the level of transport theory, should continue to be very rapid. There is considerably less to be expected from the impenetrable dense gas extension. The rarefied gas dynamics of two-point and higher Boltzmann equations should begin a most fruitful study.

Appendix: Proof that molecular chaos is equivalent to reproducibility of $f^{(1)}$.

Consider a six-dimensional domain, D, in the phase space of one molecule, $P = (x,v)$. Let $N(P_1....P_n)$ be the number of molecules with centers in D,

$$\langle N \rangle = n \int_D f^{(1)} \tag{A.1}$$

The identity

$$\left\langle (N-\langle N \rangle)^2 \right\rangle = n(n-1)\left\{ \int_{D\times D} f^{(2)} - \left(\int_D f^{(1)} \right)^2 \right\} + n\left\{ \left(\int_D f^{(1)} \right) \left(1 - \int_D f^{(1)} \right) \right\} \tag{A.2}$$

shows that the relative dispersion of N about its mean is small if and only if $f^{(2)} \sim f^{(1)} f^{(1)}$, provided only that the expected number of molecules in D, $\langle N \rangle$, is large. The cross correlation,

$$\left\langle (N-\langle N \rangle)(N'-\langle N' \rangle) \right\rangle \tag{A.3}$$

where N' is the number of molecules in D', is small whenever the self correlation, (A.2), is small for both D and D'.

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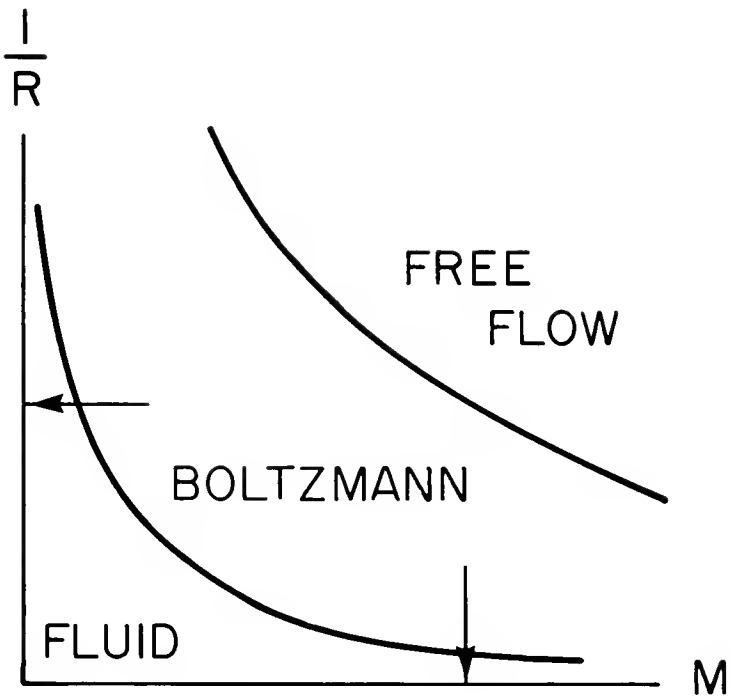

fig. I Flow Regimes

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A survey is made of various singular limits of solutions of the Boltzmann equation which, taken together, describe most of the qualitative and quantitative properties of a rarefied gas. (A more detailed survey along similar lines was presented in a Symposium in New York City in April, 1967 and published in Ref. [1].) Included are large and small mean-free-path limits, boundary and initial layers, and the influence of instabilities on the propagation of correlations by non-chaotic Boltzmann equations.

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